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On the numerical solution of the Dirichlet initial boundary-value problem for the heat equation in the case of a torus

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Abstract. The initial-boundary-value problem for the heat equation in the case of a toroidal surface with Dirichlet boundary conditions is considered. This problem is reduced to a sequence of elleptic boundary-value problems by a Laguerre transformation. The special integral representation leads to boundary-integral equations of the first kind and the toroidal surface gives one-dimensional integral equations with a logarithmic singularity. The numerical solution is realized by a trigonometric quadrature method in cases of open or closed smooth boundaries. The results of some numerical experiments are presented.

Key words: heat equation, Laguerre polynomials, integral equations of the first kind, elliptic integrals, trigonometric collocation

1. Introduction

During the last few years the integral-equation method has gained a dominant position in many numerical methods for various boundary-value problems. The boundary-element method [1] in particular is used very often. Here the advantages of integral equations and the finite-element method are combined. At the same time it is interesting to use the integral-equation method for problems with a parametric representation of the boundary. In this case we can make a large number of analytical transformations and as a result the numerical solution is simpler, with a higher order of convergence. One way of obtaining a numerical solution is to apply the quadrature method for the ensuing integral equations. This approach was successfully used in [2–6] for various boundary-value problems in the plane case. Note that in [7] this procedure is extended for boundary-value problems in $\mathbb{R}^3$.

The integral-equation method can also be effectively used for non-stationary problems. Various forms of this procedure can be applied directly by use of the time-boundary-integral representation [8, 9] or after the semi-discretization of the non-stationary problem with respect to the time variable [10, 11, 12].

In this paper we use a combination of the Laguerre transformation and the boundary-integral equation method for the heat equation in the case of a toroidal boundary. First this problem is considered in a three-dimensional spatial domain and for a special form of the boundary. Additional assumptions for the boundary function give us the possibility to obtain the corresponding integral equations on the curve in $\mathbb{R}^2$. Then we can apply the corresponding numerical methods for one-dimensional integral equations.

The outline of our paper is as follows. In Section 2 we reduce the Dirichlet initial-boundary-value problem by a Laguerre transformation to a sequence of boundary-value problems for a

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Helmholtz equation with a purely imaginary number. Next we construct an integral approach and arrive at sequences of boundary-integral equations of the first or the second kind. In Section 3 we describe the numerical solution of the integral equations by a trigonometrical quadrature method. Here distinguish the cases of a smooth open boundary and a smooth closed boundary. Section 4 presents results of numerical experiments.

Before the realization of this plan we have to formulate our non-stationary problem. Let $D \subset \mathbb{R}^3$ be an unbounded domain such that its complement is bounded and simply connected and assume that the boundary Σ of D is of class C^2. We consider the initial-boundary-value problem for the heat equation

$$\frac{1}{c}\frac{\partial u}{\partial t} = \Delta u \quad \text{in } D \times (0, \infty) \tag{1.1}$$

with heat conduction coefficient $c > 0$. We are looking for a classical solution of (1.1) that is twice continuously differentiable with respect to the space variable and continuously differentiable with respect to the time variable on $D \times (0, \infty)$ and satisfies the homogeneous initial condition

$$u(x, 0) = 0, \quad x \in D \tag{1.2}$$

and the boundary condition

$$u = F \quad \text{on } \Sigma \times [0, \infty), \tag{1.3}$$

where F is a given sufficiently smooth function satisfying the compatibility condition

$$F(x, 0) = 0, \quad x \in \Sigma.$$

At infinity we assume that

$$u(x, t) \to 0, \quad |x| \to \infty, \tag{1.4}$$

uniformly with respect to all directions $x/|x|$ and all $t \in [0, \infty)$. The questions about the existence and the uniqueness of a solution for this initial-boundary-value problem have been dealt with (see [13, Chapter 3] and [14, Chapter 9]).

2. Semi-discretization and the integral-equation method

For the reduction of the dimension in the non-stationary problem (1.1)–(1.4) we propose to use the Laguerre transformation with respect to the time variable. This transformation is successfully used in cases of hyperbolic and parabolic problems [10, 11, 12]. Thus we look for a solution of (1.1)–(1.4) in the form of a Fourier–Laguerre expansion in time

$$u(x, t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t), \tag{2.1}$$

where

$$u_n(x) = \int_0^{\infty} \mathrm{e}^{-\kappa t} L_n(\kappa t) u(x, t)\, \mathrm{d}t, \quad n = 0, 1, 2, \dots, \tag{2.2}$$

where L_n denotes the Laguerre polynomial of order n [15] and $\kappa > 0$ is a fixed parameter. The formulas (2.1) and (2.2) are interpreted as inverse and direct Laguerre transformations for the function u, respectively. Simple calculations give us the following result (for details see [11]):

Theorem 2.1 *The expansion* (2.1) *is the solution of* (1.1)–(1.4) *if and only if the coefficients* u_n *solve the sequence of boundary-value problems*

$$\Delta u_n - \gamma^2 u_n = \beta \sum_{m=0}^{n-1} u_m \quad \text{in } D, \tag{2.3}$$

$$u_n = f_n \quad \text{on } \Sigma, \tag{2.4}$$

$$u_n(x) \to 0, \quad |x| \to \infty, \tag{2.5}$$

uniformly for all directions. Here f_n *are the Fourier–Laguerre coefficients of the function* F, $\beta = \frac{\kappa}{c}$ *and* $\gamma^2 = \beta$.

Next we construct singular solutions Φ_n of (2.3) that satisfy the relations

$$\Delta \Phi_n(x, y) - \beta \sum_{m=0}^{n} \Phi_m(x, y) = \delta(|x - y|)$$

with the δ Dirac function. The functions Φ_n can be found by reduction of (2.3) to ordinary differential equations [11] or by applying the Laguerre transformation to the fundamental solution of the heat equation in $\mathbb{R}^3$ [12]. In both cases we have the following representation:

$$\Phi_n(x, y) = \frac{\mathrm{e}^{-\gamma|x-y|}}{|x - y|} \sum_{m=0}^{n} a_{n,m} |x - y|^m, \tag{2.6}$$

where the coefficients are recursively defined by

$$a_{n,0} = 1, \quad a_{n,n} = -\frac{\beta}{2\gamma n}\, a_{n-1,n-1},$$

$$a_{n,k} = \frac{1}{2\gamma k}[k(k+1)a_{n,k+1} - \beta \sum_{m=k-1}^{n-1} a_{m,k-1}], \quad k = n-1, \dots, 1,$$

for $n = 1, 2, \dots$.

Now we introduce for Equation (2.3) the single-layer potential

$$U_n(x) = \frac{1}{2\pi} \sum_{m=0}^{n} \int_\Sigma q_m(y) \Phi_{n-m}(x, y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^3 \setminus \Sigma \tag{2.7}$$

and the double-layer potential

$$V_n(x) = \frac{1}{2\pi} \sum_{m=0}^{n} \int_\Sigma q_m(y)\, \frac{\partial}{\partial \nu(y)}\, \Phi_{n-m}(x, y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^3 \setminus \Sigma \tag{2.8}$$

with continuous densities q_n for $n = 0, 1, 2, \ldots$ and the outward unit normal ν to the boundary Σ. From (2.6) we see that our singular solutions have the form

$$\Phi_n(x, y) = \frac{1}{|x - y|} + \tilde{\Phi}_n(x, y)$$

with continuous functions $\tilde{\Phi}$. Then the potentials (2.7) and (2.8) have regular properties analogous to potentials of the Laplace equation [14, Chapter 6] and we have the following theorem:

Theorem 2.2 *The single-layer potential U_n given by* (2.7) *solves the sequence of boundary-value problems* (1.1)–(1.4) *provided the densities solve a sequence of integral equations of the first kind:*

$$\frac{1}{2\pi}\int_\Sigma q_n(y)\Phi_0(x, y)\,\mathrm{d}s(y) \quad = f_n(x) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_\Sigma q_m(y)\Phi_{n-m}(x, y)\,\mathrm{d}s(y), \quad x \in \Sigma, \tag{2.9}$$

for $n = 0, 1, 2, \ldots$. The double-layer potential V_n given by (2.8) *solves the sequence of boundary-value problems* (1.1)–(1.2) *provided the densities solve a sequence of integral equations of the second kind*:

$$\begin{aligned} & q_n(x) + \frac{1}{2\pi}\int_\Sigma q_n(y)\,\frac{\partial}{\partial\nu(y)}\,\Phi_0(x, y)\,\mathrm{d}s(y) \\ &= f_n(x) - \sum_{m=0}^{n-1} q_m(x) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_\Sigma q_m(y)\,\frac{\partial}{\partial\nu(y)}\,\Phi_{n-m}(x, y)\,\mathrm{d}s(y), \quad x \in \Sigma, \end{aligned} \tag{2.10}$$

for $n = 0, 1, 2, \ldots$.

3. Numerical solution of the integral equations

In this section we focus our attention only on the sequence of integral equations of the first kind (2.9). The main reason for this is the possibility of also using these results in the case of an open boundary.

3.1. Reduction to one-dimensional equations

Let Σ be a toroidal surface, *i.e.* it is formed by rotation of some curve Γ around the axis Oz and the boundary functions f_n are axially symmetric. Then we can introduce cylindrical coordinates (r, ϕ, z). Let the boundary curve Γ be given by

$$\Gamma = \{x(s) = (r(s), z(s)) : 0 \le s \le 2\pi\},$$

where $r(s) > 0$ for all s and $x : \mathrm{IR} \to \mathrm{IR}^2$ is 2π–periodic with $|x'(s)| > 0$ for all s, in such a way that the orientation of Γ is counter-clockwise. Clearly, the torus surface Σ is given by

$$\Sigma = \{x(s, \varphi) = (r(s)\cos\varphi, r(s)\sin\varphi, z(s)) : 0 \le s, \varphi \le 2\pi\}.$$

Let boundary functions f_n also have rotational symmetry. Then we can transform the integral equations (2.9) into the following parametric form:

$$\frac{1}{2\pi}\int_0^{2\pi}\psi_n(\sigma)H_0(s,\sigma)\,\mathrm{d}\sigma = g_n(s) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi}\psi_m(\sigma)H_{n-m}(s,\sigma)\,\mathrm{d}\sigma,$$

where we have set

$$\psi_n(s) = |x'(s)|\,q_n(x(s)),\quad g_n(s) = f_n(x(s))$$

and where the kernels are given by

$$H_n(s,\sigma) = r(\sigma)\hat{\Phi}_n(x(s),x(\sigma))$$

for $s \neq \sigma$ and $n = 0, 1, \ldots$. Here we introduce singular solutions for an axially symmetric case as

$$\hat{\Phi}_n(x(s),x(\sigma)) = \int_0^{2\pi}\Phi_n(R(s,\sigma,\varphi))\,\mathrm{d}\varphi, \tag{3.1}$$

with

$$R(s,\sigma,\varphi) = ([r(s)]^2 - 2r(s)r(\sigma)\cos\varphi + [r(\sigma)]^2 + [z(s)-z(\sigma)]^2)^{\frac{1}{2}}.$$

Equation (3.1) and a Taylor expansion for the exponential function yield the representation

$$\hat{\Phi}_n(x(s),x(\sigma)) = \sum_{k=0}^{\infty}\alpha_{kn}I_k(s,\sigma),$$

where

$$\alpha_{kn} = \begin{cases}\displaystyle\sum_{j=0}^{n} a_{n,n-j}\frac{(-\gamma)^{k-n+j}}{(k-n+j)!}, & k \geq n,\\ \displaystyle\sum_{j=0}^{k} a_{n,k-j}\frac{(-\gamma)^{j}}{(j)!}, & k < n\end{cases}$$

and

$$I_n(s,\sigma) = \int_0^{2\pi}[R(s,\sigma,\varphi)]^{n-1}\,\mathrm{d}\varphi,\quad n = 0, 1, \ldots . \tag{3.2}$$

In [16] it is shown that the functions I_n satisfy the recurrence relations

$$I_{n+2} = pI_n - qJ_n,\quad (n+3)J_{n+2} = (n+1)(pJ_n - qI_n). \tag{3.3}$$

Here we have the set $p(s,\sigma) = [r(s)]^2 + [r(\sigma)]^2 + [z(s)-z(\sigma)]^2$, $q(s,\sigma) = 2r(s)r(\sigma)$ and

$$J_n(s,\sigma) = \int_0^{2\pi}\cos\varphi[R(s,\sigma,\varphi)]^{n-1}d\varphi,\quad n = 0, 1, \ldots \tag{3.4}$$

From (3.2), (3.1) and (3.3) it follows that it is necessary to distinguish the cases of even and odd indices n. Thus, for the odd case we use the formulas (3.3) with starting values $I_1 = 2\pi$ and $J_1 = 0$. For n even we start in (3.3) with the following terms

$$I_0 = \frac{4}{(p+q)^{\frac{1}{2}}}K(k)\quad\text{and}\quad J_0 = \frac{4}{q(p+q)^{\frac{1}{2}}}\left[pK(k) - (q+p)E(k)\right],$$

where K and E denote complete elliptic integrals [15] and $k^2 = \frac{2q}{p+q}$. Clearly, then for even n we have the relations

$$I_n = I_n^E E(k) + I_n^K K(k) \quad \text{and} \quad J_n = J_n^E E(k) + J_n^K K(k),$$

where I_n^E, I_n^K, J_n^E and J_n^K satisfy (3.3) with $I_0^E = 0$, $I_0^K = 4(p+q)^{-\frac{1}{2}}$, $J_0^E = -4q^{-1}(p+q)^{\frac{1}{2}}$ and $J_0^K = 4p\,q^{-1}(p+q)^{-\frac{1}{2}}$.

As a result we rewrite the functions $\hat{\Phi}_n$ in the form

$$\hat{\Phi}_n(x(s), x(\sigma)) = Q_n^E(x(s), x(\sigma))E(k) + Q_n^K(x(s), x(\sigma))K(k) + Q_n(x(s), x(\sigma))$$

with

$$Q_n^\ell(x(s), x(\sigma)) = \sum_{k=0}^{\infty} \alpha_{2k,n} I_{2k}^\ell(s, \sigma), \quad \text{for} \quad \ell = E, K \tag{3.5}$$

and

$$Q_n(x(s), x(\sigma)) = \sum_{k=0}^{\infty} \alpha_{2k+1,n} I_{2k+1}(s, \sigma). \tag{3.6}$$

Now we use the following representation for elliptic integrals [15]

$$E(k) = E_1(\eta) \log \frac{1}{\eta} + E_2(\eta) \quad \text{and} \quad K(k) = K_1(\eta) \log \frac{1}{\eta} + K_2(\eta).$$

Here $\eta = 1 - k^2$ and the functions K_ℓ and E_ℓ, $\ell = 1, 2$ have a power-series representation. But as these series are slowly convergent for some parameters k, we use for these functions the following polynomial Chebyshev approximations [17]

$$K_i(\eta) = \sum_{m=0}^{NK} a_{m,i}\eta^m \quad \text{and} \quad E_i(\eta) = \sum_{m=0}^{NE} b_{m,i}\eta^m, \quad i = 1, 2,$$

where $a_{m,i}$ and $b_{m,i}$ are the given coefficients. Note that for $NK = 10$ and $NE = 10$ the maximum absolute error for these approaches is of the order of 10^{-18}.

Therefore we can finally split the singular functions (3.1) in the form

$$\hat{\Phi}_n(x(s), x(\sigma)) = \Psi_n^1(x(s), x(\sigma)) \log \frac{1}{\eta} + \Psi_n^2(x(s), x(\sigma)), \tag{3.7}$$

where

$$\Psi_n^1(x(s), x(\sigma)) = Q_n^E(x(s), x(\sigma))E_1(\eta) + Q_n^K(x(s), x(\sigma))K_1(\eta)$$

and

$$\Psi_n^2(x(s), x(\sigma)) = Q_n^E(x(s), x(\sigma))E_2(\eta) + Q_n^K(x(s), x(\sigma))K_2(\eta) + Q_n(x(s), x(\sigma)).$$

As can be seen from (3.5) and (3.6), the functions Ψ_n^k, $k = 1, 2$ are smooth for all $s, \sigma \in [0, 2\pi]$.

Now from the representation (3.7) we can write the kernels H_n in the form

$$H_n(s, \sigma) = \log\left(\frac{4}{e}\sin^2\frac{s - \sigma}{2}\right) H_n^1(s, \sigma) + H_n^2(s, \sigma),$$

where

$$H_n^1(s,\sigma) = r(\sigma)\Psi_n^1(x(s), x(\sigma))$$

and

$$H_n^2(s,\sigma) = H_n(s,\sigma) - \log\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) H_n^1(s,\sigma)$$

for $s \neq \sigma$. The evaluation of the diagonal terms for H_n^1 is not problematic and the diagonal terms of H_n^2 are given as

$$H_n^2(s,s) = H_n^1(s,s)\log\frac{4r^2(s)}{e|x'(s)|^2} + r(s)\Psi_n^2(x(s), x(s))$$

for $n = 0, 1, \ldots$. Thus we have arrived at a sequence of integral equations of the first kind with logarithmic singularity

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\log\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) H_0^1(s,\sigma) + H_0^2(s,\sigma)\right]\psi_n(\sigma)\,\mathrm{d}\sigma = G_n(s) \tag{3.8}$$

for $0 \le s \le 2\pi$ with right-hand sides

$$G_n(s) = g_n(s) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi}\left[\log\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) H_{n-m}^1(s,\sigma) + H_{n-m}^2(s,\sigma)\right]\psi_m(\sigma)\,\mathrm{d}\sigma. \tag{3.9}$$

The uniqueness of the solutions of (3.8) is deduced from the uniqueness of the solutions of the boundary-value problems (2.3)–(2.5). By incomplete transformation we can reduce the integral equations (3.8) to a sequence of operator equations of the second kind. Then existence of a solution ψ_n in Hölder space $C^{0,\alpha}[0,2\pi]$ for any non-homogeneity $g_n \in C^{1,\alpha}[0,2\pi]$ follows from the Riesz-Schauder theory for operator equations of the second kind with a compact operator [14, Chapter 3]. Thus we have the following result:

Theorem 3.1 *For any g_n in $C^{1,\alpha}[0,2\pi]$ the integral equations* (3.8) *have unique solutions ψ_n in $C^{0,\alpha}[0,2\pi]$. Furthermore, the solutions depend continuously on g_n.*

3.2. Trigonometric quadrature method

For the numerical solution of our integral equations we combine the quadrature and collocation methods based on trigonometric interpolation with equidistant grid points. This method was suggested and analyzed in [2], including an error and convergence analysis. Now, using the trigonometric quadrature formulas for 2π-periodic functions [14, Chapter 12]

$$\frac{1}{2\pi}\int_0^{2\pi} f(s)\,\mathrm{d}s \approx \frac{1}{2M}\sum_{i=0}^{2M-1} f(s_i), \quad s_i = \frac{i\pi}{M}, \quad i = 0, 1\ldots, 2M-1, \quad M \in \mathbb{N}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\log\left(\frac{4}{e}\sin^2\frac{s_j-\sigma}{2}\right)\mathrm{d}\sigma \approx \sum_{k=0}^{2M-1} R_{|k-j|}\, f(s_k)$$

with the weights

$$R_j = -\frac{1}{2M}\left\{1 + 2\sum_{m=1}^{M-1}\frac{1}{m}\cos ms_j + \frac{(-1)^j}{M}\right\}$$

together with the collocation at the nodal points, we arrive at the following sequence of systems of linear equations

$$\sum_{k=0}^{2M-1}\psi_{n,M}(s_k)\left\{R_{|j-k|}H_0^1(s_j,s_k) + \frac{1}{2M}\ H_0^2(s_j,s_k)\right\} = G_{n,M}(s_j), \quad j = 0,\ldots,2M-1$$

with right-hand sides corresponding to (3.9).

The convergence and error analysis of this quadrature method were established in [2] on the basis of collective compact operator theory. We note only that the numerical method in [2] requires the implementation of the factor $\sin^2\frac{s-\sigma}{2}$ in the kernel H_0^1. As is shown in [18], this decomposition is not necessary for performing the error analysis. Thus from [2, 18] we have the following result:

Theorem 3.2 *Let* $\Gamma \in C^\infty$ *and* $g \in C^{p+1}[0,2\pi]$, $p \in \mathrm{IN}$. *Then the numerical solutions* $\psi_{n,M}$ *converge with respect to the Hölder norm to exact solutions* ψ_n *for* $M \to \infty$ *and for every* $n = 0,1,\ldots,N$ *and the following error estimates hold*

$$||\psi_{n,M} - \psi_n||_{0,\alpha} \le C_n M^{-p}.$$

It is clear that the numerical solution of the initial-boundary-value problem (1.1)–(1.4) has the form

$$u_{N,M}(x,t) = \frac{\kappa}{M}\sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{j=0}^{2M-1}\psi_{m,M}(s_j)r(s_j)\hat{\Phi}_{n-m}(x,x(s_j))L_n(\kappa t).$$

3.3. Smooth open-boundary case

Now we assume that the toroidal surface Σ is formed by rotation of the smooth open curve Γ with parametric representation

$$\Gamma = \{x(s) = (r(s), z(s)) : -1 \le s \le 1\},$$

where $r(s) > 0$ and $|x'(s)| > 0$ for all s and with end points $x_{-1} = x(-1)$ and $x_1 = x(1)$. Note that in this case by similar arguments as in [4] we do not assume any additional edge conditions for the solutions (2.3)–(2.5). Then the simple-layer-potential approach gives us the sequence of integral equations (2.9) and we obtain the following after parametrization

$$\frac{1}{2\pi}\int_{-1}^{1}\mu_n(\sigma)\mathcal{H}_0(s,\sigma)\mathrm{d}\sigma = g_n(s) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_{-1}^{1}\mu_m(\sigma)\mathcal{H}_{n-m}(s,\sigma)\mathrm{d}\sigma, \tag{3.10}$$

where

$$\mu_n(s) = |x'(s)|\,q_n(x(s)), \quad g_n(s) = f_n(x(s)), \quad \mathcal{H}_n(s,\sigma) = r(\sigma)\hat{\Phi}_n(x(s),x(\sigma))$$

for $n = 0,1,\ldots,N$. For the numerical solution of these integral equations we need to take into account the singularities in the densities of the form

$$\mu_n(s) = \frac{\tilde{\mu}_n(s)}{\sqrt{1-s^2}} \quad \text{for} \quad -1 < s < 1 \quad \text{and} \ \tilde{\mu}_n \in C[-1,1].$$

One of the various possibilities for the consideration of these root singularities is the variable substitution method. We use the cos-substitution according to [4, 19]. Let in (3.10) $s = \cos\zeta$ and $\sigma = \cos\varrho$. Then the simple transformations as in [4] give us from (3.10) the integral equations

$$\frac{1}{2\pi}\int_0^{2\pi} \omega_n(\varrho)\mathbf{H}_0(\zeta,\varrho)\mathrm{d}\varrho = h_n(\zeta) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi} \omega_m(\varrho)\mathbf{H}_{n-m}(\zeta,\varrho)\mathrm{d}\varrho, \tag{3.11}$$

where $\omega_n(\zeta) = \tilde{\mu}_n(\cos\zeta)$, $h_n(\zeta) = 2g_n(\cos\zeta)$ and $\mathbf{H}_n(\zeta,\varrho) = \mathcal{H}_n(\cos\zeta,\cos\varrho)$. Similarly as in the closed-boundary case we transform the kernels into the form

$$\mathbf{H}_n(\zeta,\varrho) = \mathbf{H}_n^1(\zeta,\varrho)\log\left(\frac{4}{e}\sin^2\frac{\zeta-\varrho}{2}\right) + \mathbf{H}_n^2(\zeta,\varrho)$$

with corresponding smooth functions $\mathbf{H}_n^1$ and $\mathbf{H}_n^2$. The well-posedness of Equations (3.11) in Hölder space of even functions can be shown as before (for details see [4]). The numerical solution of integral equations (3.11) and error and convergence analysis is analogous to Section 3.2. We note that the numerical values of the even densities are computed only on the interval $[0,\pi]$. Then the sequence of the linear system has, in this case, the form

$$\omega_{n,M}(0)\left[R_i\mathbf{H}_0^1(s_i,0) + \frac{1}{4M}\mathbf{H}_0^2(s_i,0)\right] + \omega_{n,M}(\pi)\left[R_{M-i}\mathbf{H}_0^1(s_i,\pi) + \frac{1}{4M}\mathbf{H}_0^2(s_i,\pi)\right] +$$
$$\sum_{j=1}^{M-1}\omega_{n,M}(s_i)\left[R_{|i-j|}\mathbf{H}_0^1(s_i,s_j) + R_{i+j}\mathbf{H}_0^1(s_i,s_{2M-j}) + \frac{1}{2M}\mathbf{H}_0^2(s_i,s_j)\right] = \mathbf{G}_n(s_j)$$

with $j = 0,\ldots,M$ and corresponding right sides $\mathbf{G}_n$.

4. Numerical experiments

4.1. Closed boundary

Assume that the torus surface Σ_1 is formed by rotation of the curve Γ_1 (see Figures 1 and 2) with parametric representation

$$\Gamma_1 = \{x(s) = (0{\cdot}2\cos s + 1,\ 0{\cdot}4\sin s - 0{\cdot}3\sin^2 s), 0 \le s \le 2\pi\}$$

and the boundary function is given by

$$F_1(x,t) = 4t^2\mathrm{e}^{-4t+2}.$$

The numerical solution of the initial boundary-value-problem (1.1)–(1.4) with $c = 1$ in two spatial points $x = (1{\cdot}5, 0)$ and $x = (1{\cdot}5, -0{\cdot}5)$ and for the time steps $t = 0\cdot0, 0\cdot5, 1\cdot5, 2\cdot0$ is presented in Table 1. Here $\kappa = 1$ and $NE = NK = 10$. The table shows the convergence of the numerical solution by an increase of values of discretization parameters. The fast convergence of the trigonometric quadrature method with respect to the number $2M$ of quadrature points is clearly exhibited. Some numerical results with discretization parameters $M = 16$

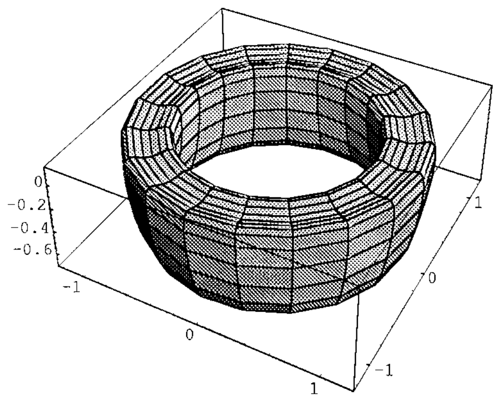

Figure 1. Torus surface Σ_1

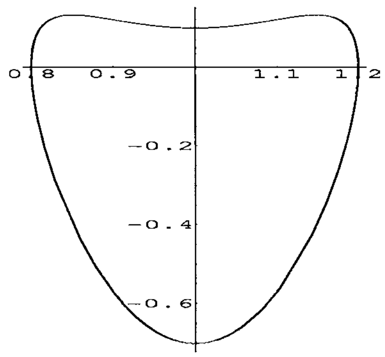

Figure 2. Rotation curve Γ_1

and $N = 30$ are presented in Figures 3 and 4. These figures show that our method works for any spatial observation points and for the selected time interval.

4.2. Open boundary

Now we consider the open torus surface Σ_2 that is formed by rotation of the half-circle with parametric representation

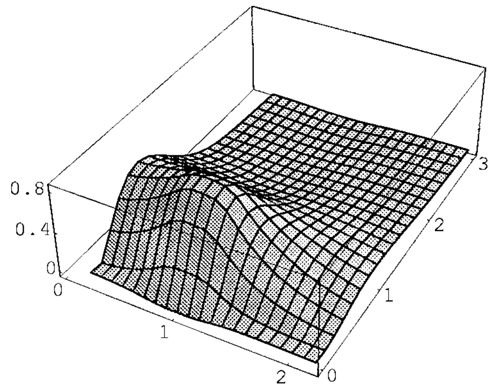

Figure 3. Numerical example 1: $z = 0{\cdot}2$, $r = 0{\cdot}2(0{\cdot}1)2{\cdot}2$, $t = 0(0{\cdot}15)3$

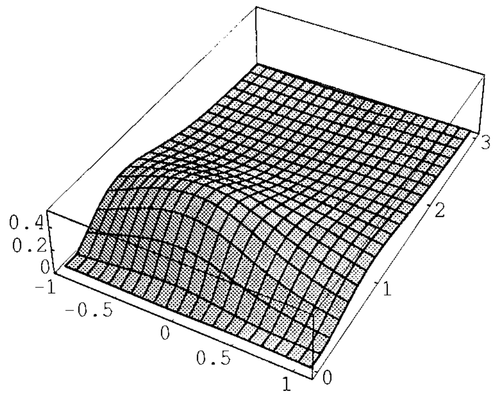

Figure 4. Numerical example 1: $r = 1{\cdot}5$, $z = -1(0{\cdot}1)1$, $t = 0(0{\cdot}15)3$

Table 1. Numerical results for the first example

		$x = (1{\cdot}5, 0{\cdot}0)$			$x = (1{\cdot}5, -0{\cdot}5)$		
t	M	$N = 30$	$N = 35$	$N = 40$	$N = 30$	$N = 35$	$N = 40$
0·0	8	0·008039	0·012936	0·012483	0·011801	0·014992	0·013517
	16	0·008036	0·012934	0·012482	0·011802	0·014993	0·013517
	32	0·008036	0·012934	0·012482	0·011802	0·014993	0·013517
0·5	8	0·474325	0·475251	0·475314	0·439599	0·440236	0·440330
	16	0·474341	0·475268	0·475331	0·439593	0·440229	0·440323
	32	0·474341	0·475268	0·475331	0·439593	0·440229	0·440323
1·0	8	0·395238	0·394595	0·394445	0·385341	0·384868	0·384518
	16	0·395239	0·394595	0·394446	0·385340	0·384867	0·384517
	32	0·395239	0·394595	0·394446	0·385340	0·384867	0·384517
1·5	8	0·177325	0·179037	0·179178	0·178542	0·179714	0·179937
	16	0·177323	0·179036	0·179176	0·178543	0·179715	0·179938
	32	0·177323	0·179036	0·179176	0·178543	0·179715	0·179938
2·0	8	0·080334	0·077963	0·077967	0·082030	0·080503	0·080799
	16	0·080332	0·077961	0·077966	0·082031	0·080503	0·080800
	32	0·080332	0·077961	0·077966	0·082031	0·080503	0·080800

$$\Gamma_2 = \{x(\zeta) = (1 + \cos\frac{\pi}{2}\zeta,\ \sin\frac{\pi}{2}\zeta),\ -1 \le \zeta \le 1\}.$$

Again, the boundary function depends only on the time and has the following form

$$F_2(x, t) = B_3\left(\frac{4t - 6}{3}\right),$$

where B_3 is the standard cubic B-spline. In the Figures 5 and 6 the numerical results of the initial-boundary-value problem (1.1)–(1.4) with these data are presented. All parameters of the method are used as in the previous examples. These results demonstrate the effectiveness

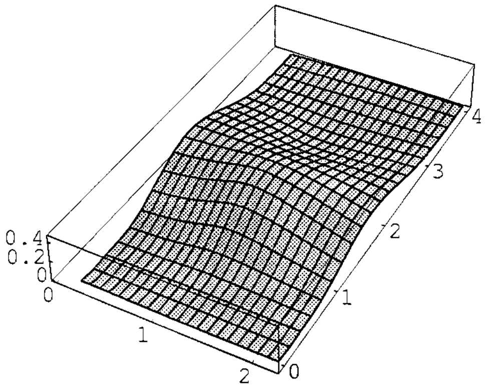

Figure 5. Numerical example 2: $z = 1{\cdot}2$, $r = 0{\cdot}2(0{\cdot}1)2{\cdot}2$, $t = 0(0{\cdot}2)4$

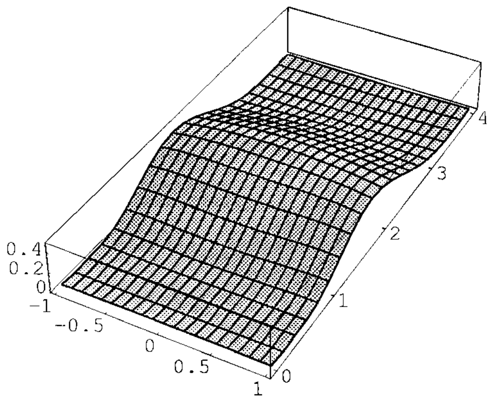

Figure 6. Numerical example 2: $r = 0{\cdot}5$, $z = -1(0{\cdot}1)1$, $t = 0(0{\cdot}2)4$

of our numerical method for the solution of time-dependent problems in unbounded domains with boundary conditions imposed on open axially symmetric surfaces.

5. Concluding remarks

The numerical solution of the time-dependent problem for the heat equation with a Dirichlet boundary condition on a toroidal surface has been considered. The proposed numerical method is based on the semi-discretization of the given problem by a Laguerre transformation with respect to the time variable and on the reduction of the pertinent steady boundary-value problems to boundary-integral equations on a surface. The axial symmetry of the boundary surface and some restrictions for the boundary function allowed us to reduce the dimension of the integral equations to one. As a result, a sequence of integral equations on the curve of a section of the torus with the logarithmic singularity in the kernels is derived. A full discretization was realized by a discrete trigonometrical collocation method. Here the quadratures that are constructed on the basis of trigonometrical interpolation for the smooth part of the integrand were used. In the case of an open smooth axially symmetric surface the singularities in the densities of the integral equations were taken into account by use of a cos-substitution. Then, integral equations similar to those of the closed-boundary case were obtained. Numerical experiments demonstrated the convergence of the proposed method.

Future investigations into this approach can be interesting for the following reasons. It is important to make a complete convergence and error analysis of the proposed method. This method works if the differential equation and initial condition are homogeneous. It is necessary to extend it to more general cases. One way would be to use a special approximation for the non-homogeneity and to construct particular solutions of the stationary differential equations [20]. In Section 2 a combination of the Laguerre transformation and the integral-equation method for a three-dimensional spatial case was developed. Clearly, the full discretization of two-dimensional integral equations requires other methods than those used for a torus. This can be realized by the boundary-element method [1] or by a spectral collocation method [7]. In future research we intend to consider these problems.

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